Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Journal of Approximation Theory 127 (2004) 198–206

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Codimension-one minimal projections onto Haar subspaces

Grzegorz Lewicki[a] and Michael Prophet[b,*]

[a] Department of Mathematics, Jagiellonian University, 30-059 Kraków, Reymonta 4, Poland
[b] Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614-0506, USA

Received 30 June 2003; accepted in revised form 2 March 2004

Communicated by E.W. Cheney

Abstract

Let H_n be an n-dimensional Haar subspace of $X = C_{\mathbb{R}}[a,b]$ and let H_{n-1} be a Haar subspace of H_n of dimension $n-1$. In this note we show (Theorem 6) that if the norm of a minimal projection from H_n onto H_{n-1} is greater than 1, then this projection is an interpolating projection. This is a surprising result in comparison with Cheney and Morris (J. Reine Angew. Math. 270 (1974) 61 (see also (Lecture Notes in Mathematics, Vol. 1449, Springer, Berlin, Heilderberg, New York, 1990, Corollary III.2.12, p. 104) which shows that there is no interpolating minimal projection from $C[a,b]$ onto the space of polynomials of degree $\leqslant n$, $(n \geqslant 2)$. Moreover, this minimal projection is unique (Theorem 9). In particular, Theorem 6 holds for polynomial spaces, generalizing a result of Prophet [(J. Approx. Theory 85 (1996) 27), Theorem 2.1].
© 2004 Elsevier Inc. All rights reserved.

MSC: 47A58; 41A05

Keywords: Haar subspace; Minimal projection; Interpolating projection

1. The results

Let X be a Banach space and let $Y \subset X$ be a closed, linear subspace. An operator $P \in \mathscr{L}(X, Y)$ is called *a projection* if $P|_Y = id_Y$. The set of all projections from X onto

*Corresponding author.
E-mail addresses: lewicki@im.uj.edu.pl (G. Lewicki), prophet@math.uni.edu (M. Prophet).

0021-9045/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2004.03.002

Y will be denoted by $\mathscr{P}(X, Y)$. A projection $P_o \in \mathscr{P}(X, Y)$ is called *minimal* if

$$||P_o|| = \lambda(Y, X) = \inf\{||P|| : P \in \mathscr{P}(X, Y)\}.$$

The constant $\lambda(Y, X)$ is called the *relative projection constant*. It is worth noting that there exists a large number of papers concerning minimal projections. Mainly the problems of existence [12,15], uniqueness [11,13,25,32,33], characterization of one-complemented subspaces [1,2,26,30,31], concrete formulas for minimal projections [3–7,10,12,21,22,24,29,34], estimates of the relative projection constants [5,14,19,23,28,31,35], construction of spaces with large relative projection constants [4,5,16–20], as well as the problems connected with shape-preserving projections [8,9] were considered. For basic information concerning this topic the reader is referred to [27].

The aim of this note is to generalize [29, Theorem 2.1] from the polynomial case to the case of Haar subspaces. More precisely, let H_n be an n-dimensional Haar subspace of $X = C_{\mathbb{R}}[a, b]$ and $H_{n-1} \subset H_n$ be an $(n-1)$-dimensional Haar space. Let $\{h_j\}_{j=1}^n \subset H_n$ be a fixed basis of H_n such that $H_{n-1} = [h_1, h_2, \ldots, h_{n-1}]$. In this note we show that for any H_n, H_{n-1} as above if $\lambda(H_{n-1}, H_n) > 1$, then a minimal projection from H_n onto H_{n-1} is an interpolating projection (Theorem 6). Moreover there is a unique minimal projection in this case (Theorem 9). We start with two well known lemmas. For sake of completeness the proofs will be included.

Lemma 1. *Let X be a Banach space and let $Y \subset X$ be a closed subspace. Suppose that $P \in \mathscr{P}(X, Y)$. Then $||P|| = 1$, if and only if for any $x \in X (Id - P)x$ is the best approximation to x in $V = ker(P)$.*

Proof. Take a linear projection P from X onto Y of norm one. Then $Id - P$ is a linear projection onto $V = ker(P)$. Take $x \in X$ and $v \in V$. Then

$$||x - (Id - P)x|| = ||x - v - (Id - P)(x - v)|| \leqslant ||P||\ ||x - v|| \leqslant ||x - v||.$$

Taking infimum over $v \in V$ we get that $dist(x, V) = ||x - (Id - P)x||$.

To prove a converse, suppose $||P|| > 1$. Then for some $x \in X$ of norm one

$$||x - (Id - P)x|| = ||Px|| > ||x|| = ||x - 0||;$$

a contradiction. □

Lemma 2. *Let X be a normed space and let $f \in X^* \backslash \{0\}$. Set $Y = ker(f)$. Then for any $P \in \mathscr{P}(X, Y)$ there exists $y_P \in X$, $f(y_P) = 1$ such that*

$$Px = x - f(x)y_P$$

for any $x \in X$.

Proof. Let $P \in \mathscr{P}(X, Y)$. Take $y_P \in ker(P)$ such that $f(y_P) = 1$. It is clear that for any $x \in X$

$$Px = x - f(x)y_P.$$

Conversely, for any $y \in X$ satisfying $f(y) = 1$ the operator $Q = Id - f(\cdot) \in \mathcal{P}(X, Y)$. $\square$

Theorem 3 (see Odyniec and Lewicki [27, p. 102]). *Let X be a finite-dimensional real Banach space and let $\mathcal{L}(X)$ denote the space of all linear operators from X into X. Assume $\mathcal{V} \subset \mathcal{L}(X)$ is a linear subspace and $L \in \mathcal{L}(X) \setminus \mathcal{V}$. Take $V_o \in \mathcal{V}$ and define*

$$E(L - V_o) = \{(x, x^*) \in S_X \times ext(S_{X^*}) : x^*(L - V_o)x = ||L - V_o||.$$

Then V_o is the best approximation to L in $\mathcal{V}$ if and only if for any $V \in \mathcal{V}$ there exists $(x, x^*) \in E(L - V_o)$ such that

$$x^*(Vx) \leqslant 0.$$

We include the following two useful lemmas about Haar spaces. These basic facts perhaps appear elsewhere but we present them with proofs for sake of completeness.

Lemma 4. *Let $Y \subset X$ be an n-dimensional Haar space. Let $y \in Y \setminus \{0\}$ have $n - 1$ distinct zeros $a \leqslant s_1 < s_2 < \cdots < s_{n-1} \leqslant b$. Then there exists $\varepsilon > 0$ such that for any $s_i \in (a, b)$ we have*

$$[s_i - \varepsilon, s_i + \varepsilon] \cap [s_j - \varepsilon, s_j + \varepsilon] = \emptyset \tag{1}$$

for $i \neq j$ and

$$y(t)y(s) < 0 \tag{2}$$

for $t \in (s_i - \varepsilon, s_i)$, $s \in (s_i, s_i + \varepsilon)$. (This implies sign changes at zeros located in the open interval (a, b).)

Proof. Since $y \in Y \setminus \{0\}$ has exactly $n - 1$ zeros (1) is obvious.

To show (2) set

$$c = \inf\left\{ |y(t)| : t \in [a, b] \Bigg\backslash \bigcup_{i=1}^{n-1} V_i \right\},$$

where for $i = 1, \ldots, n - 1$ and $V_i = (s_i - \varepsilon, s_i + \varepsilon)$. Obviously $c > 0$. Suppose, to the contrary, that there exists $j \in \{1, \ldots, n - 1\}$ such that $s_j \in (a, b)$ and

$$y(t)y(s) > 0,$$

where $t \in (s_j - \varepsilon, s_j)$ and $s \in (s_j, s_j + \varepsilon)$. Since Y is Haar there exists $z \in Y \setminus \{0\}$ such that

$$z(s_i) = \begin{cases} sgn(y(s_i - \varepsilon)) & \text{for } s_i > a, \\ z(s_i) = sgn(y(s_i + \varepsilon)) & \text{for } s_i = a. \end{cases}$$

Let $\hat{z} = \frac{cz}{2||z||}$. Note that

$$sgn((y - \hat{z})(s_j - \varepsilon)) = -sgn((y - \hat{z})(s_j)) = sgn((y - \hat{z})(s_j + \varepsilon))$$

and therefore $y - \hat{z}$ has two distinct zeros in $(s_j - \varepsilon, s_j + \varepsilon)$. Reasoning in the same way, we can show that $y - \hat{z}$ has at least one zero in each interval $(s_i - \varepsilon, s_i + \varepsilon)$, for $i \neq j$ and $s_i \in (-1, 1)$. (If $s_i = b$, we consider $(b - \varepsilon, b]$ and $[a, a + \varepsilon)$ if $s_i = a$.) Hence

$y - \hat{z}$ has at least n distinct zeros in $[a, b]$. Since $y - \hat{z} \neq 0$ this contradicts the Haar condition. This proves (2). $\square$

Lemma 5. *Let $Y \subset X$ be an n-dimensional Haar space. Then there exists $y \in Y$ such that $y(t) > 0$ for all $t \in [a, b]$.*

Proof. If $n = 1$, this is obvious. Assume $n \geqslant 2$. First we show there exists $w \in Y \setminus \{0\}$ such that $w(t) \geqslant 0$ for any $t \in [a, b]$. To accomplish this let $k \in \mathbb{N}$ and take $w_k \in Y$ such that $w_k(a) = 1$ and $w_k(t_i^k) = 0$ for

$$b - (1/k) = t_1^k < t_2^k < \cdots < t_{n-1}^k = b.$$

Set $z_k = w_k / ||w_k||$. Since the unit ball in Y is compact, we can assume that $z_k \to w \in Y$. It is clear that $||w|| = 1$ and $w(t) \geqslant 0$ on (a, b). By continuity, $w \geqslant 0$ in $[a, b]$. Note that there exists $t_o \in (a, b)$ such that $w(t_o) > 0$. Again by continuity, there exists an $\varepsilon > 0$ such that

$$c = \inf\{w(t) : t \in (t_o - \varepsilon, t_o + \varepsilon)\} > 0.$$

If n is an odd number we define $v \in Y$ by $v(a) = 1$ and $v(s_i) = 0$ where

$$t_o - \varepsilon < s_1 < s_2 < \cdots < s_{n-1} < t_o + \varepsilon.$$

By Lemma 4 and by the Haar condition $v(t) > 0$ for $t \in [a, b] \setminus (t_o - \varepsilon, t_o + \varepsilon)$. Set $\hat{v}(t) = \frac{cv}{2||v||}$. Now put $\hat{w} = w + \hat{v}$. By construction is clear that $\hat{w}(t) > 0$ for any $t \in [a, b]$.

If n is even, define $v \in Y$ by $v(a) = v(b) = 1$ and $v(s_i) = 0$ where

$$t_o - \varepsilon < s_1 < s_2 < \cdots < s_{n-2} < t_o + \varepsilon.$$

We show that also in this case $\hat{w} = w + \frac{cv}{2||v||}$ is strictly greater than 0 on $[a, b]$. Observe that $\hat{w}(t) > 0$ for any $t \in [t_o - \varepsilon, t_o + \varepsilon]$ by definition. If $\hat{w}(t) \leqslant 0$ for some $t \notin [t_o - \varepsilon, t_o + \varepsilon]$, then by the continuity of $\hat{w}$, nonnegativity of w, and definition of v $v(s_{n-1}) = 0$ for some $s_{n-1} \in (a, b) \setminus [t_o - \varepsilon, t_o + \varepsilon]$. By Lemma 4 applied to v, $v(b) < 0$, which leads to a contradiction. $\square$

Theorem 6. *For any $n \in \mathbb{N}$, let $Q_n \in \mathscr{P}(H_n, H_{n-1})$ be a minimal projection such that*

$$||Q_n|| = \lambda(H_{n-1}, H_n) > 1.$$

Then for any $n \in \mathbb{N}$, Q_n is an interpolating projection.

Proof. Since H_n is finite-dimensional, the set $\mathscr{P}(H_n, H_{n-1})$ is nonempty and a minimal projection exists (see [12,15]).

Let $Q_n \in \mathscr{P}(H_n, H_{n-1})$ be a minimal projection. Observe that any $f \in H_n$ can be represented in a unique way by

$$f = \sum_{k=1}^{n-1} a_k(f) h_k + a_n(f) h_n.$$

Hence $H_{n-1} = ker(a_n)$. By Lemma 2, there exists $y_n \in H_n$, $a_n(y_n) = 1$ such that

$$Q_n f = f - a_n(f) y_n$$

for any $f \in H_n$. Now we show that y_n has $n-1$ different zeros in $[a,b]$. Suppose this is not true. First assume that y_n is nonnegative in $[a,b]$. Set

$$\mathscr{V}_n = \{L \in \mathscr{L}(H_n, H_{n-1}) : L|_{H_{n-1}} = 0\}.$$

Since Q_n is a minimal projection $0 \in \mathscr{V}_n$ is the best approximation to Q_n in $\mathscr{V}_n$. Now take any $(x, x^*) \in E(Q_n)$. Without loss, we can assume that $x^* = t^*$ for some $t \in [a,b]$, where t^* denotes the evaluation functional at t. Hence

$$||Q_n|| = x(t) - a_n(x) y_n(t).$$

Since $||Q_n|| > 1$, $||x|| = 1$ and $y_n(t) > 0$, we have $a_n(x) < 0$. By Lemma 5 there exists $w \in P_{n-1}$ such that $w(t) > 0$ on $[a,b]$. For $f \in H_n$, set $L_n f = -a_n(f) w$. Observe that $L_n \in \mathscr{V}_n$. Since $a_n(x) < 0$, $(L_n x)t = -a_n(x) w(t) > 0$. By Theorem 3, 0 is not the best approximation to Q_n in $\mathscr{V}_n$, which leads to a contradiction.

If y_n is nonpositive in $[a,b]$, the proof goes in the same manner as above (with $L_n f := a_n(f) w$).

Now assume that $y_n(t) y_n(s) < 0$ for some $s, t \in [a,b]$. Set

$$Z = \{x \in [a,b] : y_n(x) = 0, y_n \text{ changes sign passing through } x\}.$$

From the above we have $0 < k = card(Z) < n-1$. Without loss of generality we can assume that $Z = \{x_1, \ldots, x_k\}$ and

$$a < x_1 < x_2 < \cdots < x_k < b.$$

Since $||Q_n|| > 1$ and $||(x_i)^* \circ Q_n|| = 1$, for $i = 1, \ldots, k$ there exists $\varepsilon > 0$ such that $||t^* \circ Q_n|| < ||Q_n||$ for all $t \in V = \bigcup_{i=1}^{k} (x_i - \varepsilon, x_i + \varepsilon)$ and $x_k + \varepsilon < b - \varepsilon$. Set

$$crit(Q_n) = \{t \in [a,b] : ||t^* \circ Q_n|| = ||Q_n||\}. \tag{3}$$

Since $||Q_n|| > 1$, $y_n(t) \neq 0$ for any $t \in crit(Q_n)$. Now we construct a function $q_n \in H_{n-1}$ such that

$$y_n(t) q_n(t) < 0 \tag{4}$$

for any $t \in crit(Q_n)$. Without loss, we can assume that $y_n|_{[a,x_1]} \geqslant 0$. First suppose that $n-k$ is an odd number. Set $z_n(a) = 1$, $z_n(b) = 0$, $z_n(x_i) = 0$ for $i = 1, \ldots, k$ and $z_n(u_j) = 0$ for $j = 1, \ldots, n-3-k$, if $k < n-3$. Here for $j = 1, \ldots, n-3-k$, $u_j \in [a,b]$ are so chosen that $x_k < u_1 < \cdots < u_{n-3-k} < x_k + \varepsilon$. Since H_{n-1} is an $(n-1)$-dimensional Haar space, there exists exactly one $z_n \in H_{n-1}$ satisfying the above conditions. Set

$$c = \inf\{|z_n(t)| : t \in [a,b] \backslash (V \cup (b - \varepsilon, b])\}.$$

Observe that by the compactness argument $c > 0$. By Lemma 5 there exists $w \in H_{n-1}$ such that $w(t) > 0$ for any $t \in [a,b]$. Set $w_n = \frac{cw}{2||w||}$ and $q_n = -(z_n + w_n)$. By the construction of q_n, and Lemma 4 applied to q_n, (4) is satisfied for any $t \in crit(Q_n)$.

Now assume that $n-k$ is an even number. Set $z_n(a) = 1$, $z_n(x_i) = 0$ for $i = 1, \ldots, k$ and $z_n(u_j) = 0$ for $j = 1, \ldots, n-2-k$, if $k < n-2$. Here for $j = 1, \ldots, n-2-k$,

$u_j \in [a,b]$ are so chosen that $x_k < u_1 < \cdots < u_{n-2-k} < x_k + \varepsilon$. Put $q_n = -z_n$. By the construction of q_n, and Lemma 4 applied to q_n, (4) is satisfied for any $t \in crit(Q_n)$.

Now define for $f \in H_n$

$$L_n f = a_n(f) q_n.$$

Observe that $L_n \in \mathscr{V}_n$. Now take any $(x, t^*) \in E(Q_n)$. It is obvious that $t \in crit(Q_n)$. Observe that

$$||Q_n|| = x(t) - a_n(x) y_n(t).$$

Since $||Q_n|| > 1$ and $||x|| = 1$, $a_n(x) y_n(t) < 0$. Hence $y_n(t) \neq 0$. By the definition of q_n,

$$(L_n x)t = a_n(x) q_n(t) > 0.$$

By Theorem 3, 0 is not the best approximation to Q_n in $\mathscr{V}_n$ and consequently Q_n is not a minimal projection; a contradiction. This shows that y_n has $n-1$ different roots in $[a,b]$. Let us denote them by $t_1, \ldots, t_{n-1}$. Define for $f \in H_n$

$$Z_n f = \sum_{j=1}^{n-1} f(t_j) l_j,$$

where $l_j \in H_{n-1}$ are so chosen that $l_j(t_i) = \delta_{ij}$ for $i, j = 1, \ldots, n-1$. It is clear that Z_n is an interpolating projection from H_n onto H_{n-1} and $Z_n(y_n) = 0$. Since $Q_n(y_n) = 0$, $Q_n = Z_n$, which shows our claim. $\square$

Theorem 7 (Compare with [12, Theorem 9]). *For $n \in \mathbb{N}$, let $P_n \subset C_R[a,b]$ denote the space of all polynomials of degree $\leqslant n$. Then for any $n \geqslant 1$ a minimal projection from P_n onto P_{n-1} is an interpolating projection. Moreover, $\lambda(P_{n-1}, P_n) > 1$ for $n \geqslant 3$.*

Proof. Let for $j = 0, \ldots, n$ and $t \in [a,b]$ $p_j(t) = t^j$. If $n = 1$ then the projection Q_1 from P_1 onto P_0 given by $Q_1(f) = f((a+b)/2)1$ is clearly an interpolating projection of norm one. If $n = 2$ then it is clear that the interpolating projection from P_2 onto P_1 with the nodes at a and b has norm one.

Now assume $n \geqslant 3$. Let $Q_n \in \mathscr{P}(P_n, P_{n-1})$ be a minimal projection. Observe that any $f \in P_n$ can be represented in a unique way by

$$f = \sum_{k=0}^{n-1} a_k(f) p_k + a_n(f) p_n.$$

Hence $P_{n-1} = ker(a_n)$. By Lemma 2, there exists $y_n \in P_n$, $a_n(y_n) = 1$ such that

$$Q_n f = f - a_n(f) y_n$$

for any $f \in P_n$. First we show that $||Q_n|| > 1$. Assume this is not true. Then by Lemma 1, 0 is the best approximation in $ker(Q_n) = span[y_n]$ to any $p \in P_{n-1}$. Since $a_n(y_n) = 1$, $y_n(t_o) \neq 0$ for some $t_o \in (a,b)$. Let $q_o(t) = (b-a)^2 - (t - t_o)^2$. Since $n \geqslant 3$, $q_o \in P_{n-1}$. Since $y_n(t_o) \neq 0$ it is easy to see that

$$||q_o - \alpha y_n|| < ||q_o||$$

for $\alpha \in \mathbb{R}$ sufficiently small, which leads to a contradiction. By Theorem 6 Q_n is an interpolating projection, as required. $\square$

Now we consider the problem of uniqueness of minimal projections. The proof of the next proposition is similar to that of [12, Theorem 10], (see also [28, Theorem 4.8]).

Proposition 8. *Let H_n, H_{n-1} be as in Theorem 6. Let $Q_n \in \mathcal{P}(H_n, H_{n-1})$ be a minimal projection. Then $crit(Q_n)$ (see (3)) consists of at least n points.*

Proof. Since H_n is finite-dimensional, $crit(Q_n) \neq \emptyset$. Assume that $crit(Q_n) = \{t_1, \dots, t_k\}$ where $k \leqslant n-1$. Let R_n be the interpolating projection determined by $t_1, \dots, t_k$ (if $k < n-1$ we add to $\{t_1, \dots, t_k\}$ $n-k-1$ points $s_1, \dots, s_{n-k-1}$ from $[a,b]$). Set $L_n = Q_n - R_n$. It is clear that $L_n \in \mathcal{V}_n$. Now take any $(x, t^*) \in E(Q_n)$. It is obvious that $t = t_i$ for some $i \in \{1, \dots, k\}$. Note that

$$(L_n x)t_i = (Q_n x)t_i - (R_n x)t_i = ||Q_n|| - x(t_i) \geqslant ||Q_n|| - 1 > 0.$$

By Theorem 3, 0 is not the best approximation to Q_n in $\mathcal{V}_n$, and consequently Q_n is not a minimal projection; a contradiction. $\square$

Theorem 9. *Let H_n and H_{n-1} be as in Theorem 6. Then there is exactly one minimal projection from H_n onto H_{n-1}.*

Proof. Suppose that Q_n and R_n are two different minimal projections. Then obviously, $S_n = (Q_n + R_n)/2$ is also a minimal projection. Since $||S_n|| > 1$, by Proposition 8, $crit(S_n)$ consists of at least n different points $t_1, \dots, t_n$. Since H_n is finite-dimensional, there exists $x_1, \dots, x_n \in H_n$, of norm one, such that

$$||S_n|| = (S_n x_i)t_i$$

for $i = 1, \dots, n$. Observe that for any $i \in \{1, \dots, n\}$,

$$||S_n|| = (S_n x_i)t_i = \frac{(Q_n x_i)t_i + (R_n x_i)t_i}{2} \leqslant \frac{||Q_n|| + ||R_n||}{2} = ||S_n||.$$

Hence for any $i = 1, \dots, n$,

$$(Q_n x_i)t_i = (R_n x_i)t_i = ||S_n||.$$

By Lemma 2, Q_n is determined by $y_Q \in P_n$ satisfying $a_n(y_Q) = 1$ and R_n is determined by $y_R \in P_n$ satisfying $a_n(y_R) = 1$. By the above equality, for any $i \in \{1, \dots, n\}$,

$$||S_n|| = x_i(t_i) - a_n(x_i)y_Q(t_i) = x_i(t_i) - a_n(x_i)y_R(t_i).$$

Since $||S_n|| > 1$ and $||x_i|| = 1$, $a_n(x_i) \neq 0$ for $i = 1, \dots, n$. Consequently

$$y_Q(t_i) = y_R(t_i)$$

for $i = 1, \dots, n$, which gives immediately $y_Q = y_R$. Hence $Q_n = R_n$; a contradiction. $\square$

References

[1] J. Blatter, E.W. Cheney, Minimal projections onto hyperplanes in sequence spaces, Ann. Mat. Pura Appl. 101 (1974) 215–227.
[2] H.F. Bohnenblust, Subspaces of $l_{p,n}$-spaces, Amer. J. Math. 63 (1941) 64–72.
[3] B.L. Chalmers, G. Lewicki, Minimal projections onto symmetric spaces with large projection constants, Studia Math. 134 (2) (1999) 119–133.
[4] B.L. Chalmers, G. Lewicki, Two-dimensional real symmetric spaces with maximal projection constants, Ann. Polnici Math. 72 (2) (2000) 119–134.
[5] B.L. Chalmers, G. Lewicki, Symmetric spaces with maximal projection constants, J. Funct. Anal. 200 (1) (2003) 1–22.
[6] B.L. Chalmers, F.T. Metcalf, The determination of minimal projections and extensions in L_1, Trans. Amer. Math. Soc. 329 (1992) 289–305.
[7] B.L. Chalmers, F.T. Metcalf, A characterization and equations for minimal projections and extensions, J. Operator Theory 32 (1994) 31–46.
[8] B.L. Chalmers, M. Prophet, Minimal shape-preserving projections onto π_n, Numer. Funct. Anal. Optim. 18 (1997) 507–520.
[9] B.L. Chalmers, M. Prophet, Existence of shape-preserving A-actions, Rocky Mountain J. Math. 28 (1998) 813–833.
[10] E.W. Cheney, C. Franchetti, Minimal projections in $\mathscr{L}_1$-spaces, Duke Math. J. 43 (3) (1976) 501–510.
[11] E.W. Cheney, C.R. Hobby, P.D. Morris, F. Schurer, D.E. Wulbert, On the minimal property of the Fourier projection, Trans. Amer. Math. Soc. 143 (1969) 249–258.
[12] E.W. Cheney, P.D. Morris, On the existence and characterization of minimal projections, J. Reine Angew. Math. 270 (1974) 61–76.
[13] S.D. Fisher, P.D. Morris, D.E. Wulbert, Unique minimality of Fourier projections, Trans. Amer. Math. Soc. 265 (1981) 235–246.
[14] C. Franchetti, Projections onto hyperplanes in banach spaces, J. Approx. Theory 38 (1983) 319–333.
[15] J.R. Isbell, Z. Semadeni, Projection constants and spaces of continuous functions, Trans. Amer. Math. Soc. 107 (1) (1963) 38–48.
[16] H. König, Spaces with large projection constants, Israel J. Math. 50 (1985) 181–188.
[17] H. König, Projections onto symmetric spaces, Quaest. Math. 18 (1995) 199–200.
[18] H. König, C. Schuett, N. Tomczak-Jaegermann, Projection constants of symmetric spaces and variants of Khinchine's inequality, J. Reine Angew. Math. 511 (1999) 1–42.
[19] H. König, N. Tomczak-Jaegermann, Bounds for projection constants and 1-summing norms, Trans. AMS 320 (1990) 799–823.
[20] H. König, N. Tomczak-Jaegermann, Norms of minimal projections, J. Funct. Anal. 119 (1994) 253–280.
[21] G. Lewicki, Kolmogorov's type criteria for spaces of compact operators, J. Approx. Theory 64 (1991) 181–202.
[22] G. Lewicki, Minimal projections onto two dimensional subspaces of $l_\infty^{(4)}$, J. Approx. Theory (88) (1997) 92–108.
[23] G. Lewicki, Minimal extensions in tensor product spaces, J. Approx. Theory (97) (1999) 366–383.
[24] G. Lewicki, On minimal projections in $l_\infty^{(n)}$, Monatsh. Math. (129) (2000) 119–131.
[25] G. Lewicki, G. Marino i, P. Pietramala, Fourier-type minimal extensions in real L_1-spaces, Rocky Mountain J. Math. 30 (3) (2000) 1025–1037.
[26] J. Lindenstrauss, On projections with norm one-an example, Proc. Amer. Math. Soc. 15 (3) (1964) 403–406.
[27] Wł. Odyniec, G. Lewicki, Minimal projections in Banach Spaces, in: Lecture Notes in Mathematics, Vol. 1449, Springer, Berlin, Heilderberg, New York, 1990.
[28] K.C. Pan. B. Shekhtman, On minimal interpolating projections and trace duality, J. Approx. Theory 65 (2) (1991) 216–230.
[29] M. Prophet, Codimension one minimal projections onto the quadratics, J. Approx. Theory 85 (1996) 27–42.

[30] B. Randrianantoanina, One-complemented subspaces of real sequence spaces, Results Math. (33) (1998) 139–154.
[31] S. Rolewicz, On minimal projections of the space L ([0,1]) on one-codimensional subspace, Bull. Acad. Polon. Sci. Math. 34 (3–4) (1986) 151–153.
[32] L. Skrzypek, The uniqueness of norm-one projection in James-Type spaces, J. Approx. Theory 100 (1997) 73–93.
[33] L. Skrzypek, Uniqueness of minimal projections in smooth matrix spaces, J. Approx. Theory (107) (2000) 315–336.
[34] L. Skrzypek, Minimal projections in spaces of functions of n variables, J. Approx. Theory 123 (2) (2003) 214–231.
[35] D.E. Wulbert, Some complemented function spaces in $C(X)$, Pacific J. Math. 24 (3) (1968) 589–602.